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On the minimum of the Christoffel function

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Abstract

We consider the Christoffel function $\lambda_n(\mathrm{d}\alpha)$ associated with a measure $\mathrm{d}\alpha$ on a compact interval and the question where the minimum of the Christoffel function is attained. Sufficient conditions on the coefficients appearing in the continued fraction expansion of the Stieltjes transform are obtained for the nth Christoffel function to attain its minimum at the boundary points of the underlying interval. In contrary to several monotonicity results for Christoffel functions in the literature, our approach does not require the absolute continuity of the measure α with respect to the Lebesgue measure.

Keywords: Christoffel function; Weighted approximation; Orthogonal polynomials; Continued fractions; Stieltjes transform

AMS Classification: 33C45; 41A44

1. Introduction

Let α denote a probability measure on a compact interval, say $[a, b]$ $(a < b)$, and consider the nth Christoffel function

$$\lambda_n(\mathrm{d}\alpha, x) = \min \left\{ \int_a^b |\Pi(t)|^2 \, \mathrm{d}\alpha(t) \,|\, \Pi \in \mathbb{P}_{n-1}, \Pi(x) = 1 \right\}, \tag{1}$$

where $\mathbb{P}_{n-1}$ is the set of all polynomials of degree $n-1$. The concept of Christoffel functions plays a crucial role in weighted approximation and orthogonal polynomials. Lower bounds for $\lambda_n(\mathrm{d}\alpha, x)$ are important for proving Jackson theorems, and in estimating spacings of zeros of orthogonal polynomials. For more details, the interested reader is referred to the work in [14]. In this paper, we investigate the question whether the minimum of the nth Christoffel function $\lambda_n(\mathrm{d}\alpha, x)$ over an interval $[\tilde{a}, \tilde{b}] \supseteq [a, b]$ is attained at the boundary point $\tilde{a}$ or $\tilde{b}$. A similar problem appears in the

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area of mathematical statistics and has been partially solved in [5]. The present paper has three tasks: first, we generalize the results in [5], secondly, we want to make the theory accessible to approximators and finally, we demonstrate its application for several probability measures which might be of interest in approximation theory but not in statistics. Section 2 introduces the necessary notation. Section 3 investigates conditions such that the orthogonal polynomials with respect to the measure $\mathrm{d}\alpha$ attain their maximal absolute value at the boundary points of the interval $[\tilde{a},\tilde{b}] \supseteq [a,b]$. Finally, in Section 4, sufficient conditions are given such that the Christoffel function has its minimum at the point $\tilde{a}$ or $\tilde{b}$. All results are based on properties of the coefficients in the continued fraction expansion for the Stieltjes transform of the measure $\mathrm{d}\alpha$. The theory is illustrated for the distributions corresponding to the Jacobi-, associated Pollaczek-, Tricomi–Carlitz polynomials and a system of sieved orthogonal polynomials introduced in [7].

2. Preliminaries

Consider the Stieltjes transform of the probability measure α on a compact interval $[a,b]$ and its corresponding continued fraction expansion

$$\int_a^b \frac{\mathrm{d}\alpha(t)}{z-t} = \frac{1}{|z-a-(b-a)\zeta_1} - \frac{(b-a)^2\zeta_1\zeta_2}{|z-a-(b-a)(\zeta_2+\zeta_3)} - \frac{(b-a)^2\zeta_3\zeta_4}{|z-a-(b-a)(\zeta_4+\zeta_5)} - \cdots, \tag{2}$$

where $\zeta_1 = p_1$, $\zeta_j = q_{j-1}p_j$ $(j \geqslant 2)$, $q_j = 1 - p_j$ and $p_j \in [0,1]$ (see e.g. [12]). It is well known that the polynomial in the denominator of the nth convergent in (2) defines the nth monic orthogonal polynomial $\Phi_n(\mathrm{d}\alpha,x)$ with respect to the measure $\mathrm{d}\alpha$, and can be calculated recursively by $\Phi_{-1}(\mathrm{d}\alpha,x)=0$, $\Phi_0(\mathrm{d}\alpha,x)=1$,

$$\Phi_{n+1}(\mathrm{d}\alpha,x) = [x-a-(b-a)(\zeta_{2n}+\zeta_{2n+1})]\Phi_n(\mathrm{d}\alpha,x) - (b-a)^2\zeta_{2n-1}\zeta_{2n}\Phi_{n-1}(\mathrm{d}\alpha,x) \tag{3}$$

$(n \geqslant 0, \zeta_0 = 0)$. The measure α has finite support if and only if the continued fraction in (2) terminates, i.e. $p_j \in \{0,1\}$ for some $j \in \mathbb{N}$ (see [17]). The nth monic orthogonal polynomial has squared L_2-norm

$$\kappa_n^{-1} = \int_a^b \Phi_n^2(\mathrm{d}\alpha,x)\,\mathrm{d}\alpha(x) = (b-a)^{2n}\prod_{j=1}^n \zeta_{2j-1}\zeta_{2j}$$

and we denote by $P_n(\mathrm{d}\alpha,x) = \sqrt{\kappa_n}\Phi_n(\mathrm{d}\alpha,x)$ the orthonormal polynomials with respect to the measure $\mathrm{d}\alpha$. The nth Christoffel function can now be represented in terms of the polynomials $P_n(\mathrm{d}\alpha,x)$ as

$$\lambda_n(\mathrm{d}\alpha,x) = \left[\sum_{j=0}^{n-1} P_j^2(\mathrm{d}\alpha,x)\right]^{-1} \quad (x \in \mathbb{R}),\ n = 1,2,\ldots. \tag{4}$$

Throughout this paper, we are interested in finding the minimum of the function $\lambda_n(\mathrm{d}\alpha,x)$ over an interval $[\tilde{a},\tilde{b}]$ which contains the interval of orthogonality of the measure α, i.e., $[a,b] \subseteq [\tilde{a},\tilde{b}]$. Observing the identity (4) we see that this problem is related to the question of finding the absolute maxima of the orthonormal polynomials $P_j(\mathrm{d}\alpha, x)$. For a Borel set B let

$$I_B(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{else,} \end{cases}$$

denote the corresponding indicator function. Obviously, the polynomials $\Phi_n(\mathrm{d}\alpha, x)$ [or equivalently $P_n(\mathrm{d}\alpha, x)$] are also orthogonal [orthonormal] on the interval $[\tilde{a}, \tilde{b}]$ ($\supseteq [a,b]$) with respect to the measure $\mathrm{d}\tilde{\alpha}(x) = I_{[a,b]}(x)\,\mathrm{d}\alpha(x)$. Thus, there exists also a recurrence relation corresponding to (3)

$$\begin{aligned} \Phi_{n+1}(\mathrm{d}\alpha, x) &= [x - \tilde{a} - (\tilde{b} - \tilde{a})(\tilde{\zeta}_{2n} + \tilde{\zeta}_{2n+1})]\,\Phi_n(\mathrm{d}\alpha, x) \\ &\quad - (\tilde{b} - \tilde{a})^2 \tilde{\zeta}_{2n-1}\tilde{\zeta}_{2n}\Phi_{n-1}(\mathrm{d}\alpha, x)\,, \end{aligned} \tag{5}$$

where $\tilde{\zeta}_1 = \tilde{p}_1$, $\tilde{\zeta}_j = \tilde{q}_{j-1}\tilde{p}_j\ (j \geqslant 2)$, $\tilde{q}_j = 1 - \tilde{p}_j$ and the quantities $\tilde{p}_j \in [0,1]$ can be determined from the numbers $p_j \in [0,1]$ by

$$\tilde{p}_1 = \frac{a - \tilde{a} + (b-a)p_1}{\tilde{b} - \tilde{a}}\,, \tag{6}$$

$$\begin{aligned} &a + (b-a)(q_{2j-1}p_{2j} + q_{2j}p_{2j+1}) = \tilde{a} + (\tilde{b} - \tilde{a})(\tilde{q}_{2j-1}\tilde{p}_{2j} + \tilde{q}_{2j}\tilde{p}_{2j+1}) \quad (j \geqslant 1)\,, \\ &(b-a)^2 q_{2j-2}p_{2j-1}q_{2j-1}p_{2j} = (\tilde{b} - \tilde{a})^2 \tilde{q}_{2j-2}\tilde{p}_{2j-1}\tilde{q}_{2j-1}\tilde{p}_{2j} \quad (j \geqslant 1)\,. \end{aligned} \tag{7}$$

Note that (6) and (7) can be solved recursively ($\tilde{q}_0 = q_0 = 0$) and that in the jth step the two equations of (7) are two linear equations for $\tilde{p}_{2j}$ and $\tilde{p}_{2j+1}$. It seems to be intractable to obtain an explicit representation of $\tilde{p}_j$ in terms of $p_1, \ldots, p_j$. However, under the additional assumptions of symmetry, the solution of (6) and (7) becomes more transparent.

Lemma 2.1. *If $a = -b$, $\tilde{a} = -\tilde{b}$ and the measure α is symmetric, then the quantities $\tilde{p}_j$ in the recurrence relation* (5) *can be calculated as*

$$\tilde{p}_{2j} = \frac{b^2 q_{2j-2}p_{2j}\Phi_{j-1}(\mathrm{d}\alpha, \tilde{b})}{\tilde{b}\Phi_j(\mathrm{d}\alpha, \tilde{b})} \quad (j \geqslant 1)\,,$$

where the polynomials $\Phi_j(\mathrm{d}\alpha, x)$ are the monic orthogonal polynomials with respect to the measure $\mathrm{d}\alpha$ *defined in* (3).

Proof. Because the measure α is symmetric on the intervals $[-b, b]$ and $[-\tilde{b}, \tilde{b}]$ we have from [11]:

$$p_{2j-1} = \tilde{p}_{2j-1} = \tfrac{1}{2} \quad (j \geqslant 1)$$

and (7) reduces to ($q_0 = \tilde{q}_0 = 1$)

$$\frac{b^2}{\tilde{b}^2}\, q_{2j-2}p_{2j} = \tilde{q}_{2j-2}\tilde{p}_{2j} \quad (j \geqslant 1)\,.$$

The proof now follows by induction observing the recurrence relation (3), which implies

$$\tilde{q}_{2j} = \frac{\Phi_{j+1}(\mathrm{d}\alpha, \tilde{b})}{\tilde{b}\Phi_j(\mathrm{d}\alpha, \tilde{b})}\,. \qquad \square$$

3. The absolute maximum of the nth orthonormal polynomial

In this section, we present a sufficient condition such that the orthonormal polynomials with respect to a given measure on $[a,b]$ attain their absolute maxima in the interval $[\tilde{a},\tilde{b}] \supseteq [a,b]$ at the point $\tilde{a}$ or $\tilde{b}$.

Theorem 3.1. *Let α be a probability measure on the interval $[a,b] \subseteq [\tilde{a},\tilde{b}]$, $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \dots$, denote the quantities in the recurrence relation (5) for the monic orthogonal polynomials with respect to the measure $\mathrm{d}\alpha$. If*

$$\tilde{p}_{2j-1} \leqslant \tfrac{1}{2}, \quad j = 1,\dots,n+1, \qquad \tilde{p}_{2j} \leqslant \tfrac{1}{2}, \quad j = 1,\dots,n, \qquad \tilde{p}_{2n+2} \in (0,1)\,,$$

then the orthonormal polynomials $P_1(\mathrm{d}\alpha,x),\dots,P_{n+1}(\mathrm{d}\alpha,x)$ with respect to the measure $\mathrm{d}\alpha$ attain their absolute maxima over the interval $[\tilde{a},\tilde{b}]$ at the point $\tilde{b}$.

Proof. Let $\mathrm{d}\tilde{\alpha}(x) = I_{[a,b]}(x)\,\mathrm{d}\alpha(x)$ and $\{Q_j\}_{j=0}^{\infty}$, $\{S_j\}_{j=0}^{\infty}$ denote the orthonormal polynomials with respect to the measures $(\tilde{b}-x)(x-\tilde{a})\,\mathrm{d}\tilde{\alpha}(x)$, $(\tilde{b}-x)\,\mathrm{d}\tilde{\alpha}(x)$. By Theorems 4.1(c) and 3.5(c) in [3], these polynomials satisfy the identities

$$\begin{aligned}(\tilde{b}-x)(x-\tilde{a})\sum_{l=0}^{k-1}\tilde{\beta}_l Q_l^2(x) + (\tilde{b}-x)(x-\tilde{a})\gamma_k Q_k^2(x) + (\tilde{b}-x)\sum_{l=0}^{k}\tilde{\delta}_l S_l^2(x)&\\ = 1 - \varepsilon_k P_{k+1}^2(\mathrm{d}\alpha,x) \quad (k=0,\dots,n)&\end{aligned} \tag{8}$$

where

$$\tilde{\beta}_l = \frac{\tilde{p}_{2l+1}}{\tilde{q}_{2l+1}}\prod_{j=1}^{l}\frac{\tilde{p}_{2j-1}\tilde{p}_{2j}}{\tilde{q}_{2j-1}\tilde{q}_{2j}}\left(1-\frac{\tilde{p}_{2l+2}}{\tilde{q}_{2l+2}}\right), \quad l = 1,\dots,k-1,$$

$$\tilde{\delta}_l = \prod_{j=1}^{l}\frac{\tilde{p}_{2j-1}\tilde{p}_{2j}}{\tilde{q}_{2j-1}\tilde{q}_{2j}}\left(1-\frac{\tilde{p}_{2l+1}}{\tilde{q}_{2l+1}}\right), \quad l = 1,\dots,k,$$

and $\gamma_k \geqslant 0$, $\varepsilon_k \geqslant 0$ $(k=0,\dots,n)$. The assumptions of the theorem imply $\tilde{\delta}_l \geqslant 0$ $(l=1,\dots,n)$, $\tilde{\beta}_l \geqslant 0$ $(l=0,\dots,n-1)$ and we obtain from (8) for all $k=0,\dots,n$,

$$P_{k+1}^2(\mathrm{d}\alpha,x) \leqslant \frac{1}{\varepsilon_k}$$

whenever $x \in [\tilde{a},\tilde{b}]$ with equality at the point $x = \tilde{b}$. □

Remark 3.2. Let α denote a measure on the interval $[a,b]$ with corresponding sequence $p_1, p_2, p_3, \dots$ in (2) and α^* the measure induced by the reflection at the point $\frac{1}{2}(a+b)$, i.e.,

$$\alpha^*(B) = \alpha(b+a-B) \quad \text{for all Borel sets } B.$$

From (2) it follows that

$$\begin{aligned}\int_a^b \frac{\mathrm{d}\alpha^*(t)}{z-t} &= \frac{-1}{|-z+b-(b-a)\zeta_1} - \frac{(b-a)^2\zeta_1\zeta_2}{|-z+b-(b-a)(\zeta_2+\zeta_3)} - \cdots\\ &= \frac{1}{|z-a-(b-a)\zeta_1^*} - \frac{(b-a)^2\zeta_1^*\zeta_2^*}{|z-a-(b-a)(\zeta_2^*+\zeta_3^*)} - \frac{(b-a)^2\zeta_3^*\zeta_4^*}{|z-a-(b-a)(\zeta_4^*+\zeta_5^*)} - \cdots\end{aligned}$$

where $\zeta_1^* = q_1$, $\zeta_{2j}^* = p_{2j-1}p_{2j}$, $\zeta_{2j+1}^* = q_{2j}q_{2j+1}$ ($j \geqslant 1$). Consequently, the quantities p_j^* corresponding to the continued fraction expansion for the Stieltjes transform of the reflected measure α^* satisfy

$$p_{2j}^* = p_{2j}, \quad p_{2j-1}^* = q_{2j-1} \qquad (j \geqslant 1)$$

(see also [16]). Applying the same argument to the interval $[\tilde{a}, \tilde{b}]$, we see from Theorem 3.1 that the inequalities $\tilde{p}_{2j-1} \geqslant \frac{1}{2}$ ($j = 1, \dots, n+1$), $\tilde{p}_{2j} \leqslant \frac{1}{2}$ ($j = 1, \dots, n$), $\tilde{p}_{2n+2} \in (0,1)$ imply that the orthonormal polynomials $P_1(\mathrm{d}\alpha, x), \dots, P_{n+1}(\mathrm{d}\alpha, x)$ with respect to the measure $\mathrm{d}\alpha$ attain their absolute maxima over the interval $[\tilde{a}, \tilde{b}]$ at the point $\tilde{a}$.

Example 3.3 (*Jacobi polynomials*). For the Jacobi polynomials $P_n^{(\alpha,\beta)}$, we have $[a,b] = [-1,1]$, $\mathrm{d}\alpha(x) = (1-x)^\alpha(1+x)^\beta\,\mathrm{d}x$ ($\alpha, \beta > -1$) and

$$p_{2j} = \frac{j}{2j+1+\alpha+\beta} \quad (j \geqslant 1),$$

$$p_{2j-1} = \frac{\beta+j}{\alpha+\beta+2j} \quad (j \geqslant 1)$$

(see [16]). Thus, if $\alpha + \beta \geqslant -1$, $\alpha \geqslant \beta$ we have $p_j \leqslant \frac{1}{2}$ $\forall j \in \mathbb{N}$ and Theorem 3.1 shows that the nth Jacobi polynomial attains its absolute maximum over the interval $[-1,1]$ at the point 1 while for $\alpha + \beta \geqslant -1$, $\alpha \leqslant \beta$, the absolute maximum is attained at -1.

Similarly, it follows from the recurrence relation of the associated ultraspherical polynomials

$$(n+v+1)P_{n+1}^{(\alpha)}(x,v) = 2x(n+v+\alpha)P_n^{(\alpha)}(x,v) - (2\alpha+n+v-1)P_{n-1}^{(\alpha)}(x,v)$$

and

$$P_0^{(\alpha)}(x,v) = 1, \qquad P_1^{(\alpha)}(x,v) = \frac{2\alpha+2v}{v+1}\,x$$

($\alpha > 0$) that the quantities p_j in the continued fraction expansion (2) are given by $p_{2j-1} = \frac{1}{2}$ and

$$p_{2j} = \frac{(v+j)(2\alpha-1+v)_{j+1} - (j+2\alpha-1+v)(v)_{j+1}}{2(j+\alpha+v)[(2\alpha+v-1)_{j+1} - (v)_{j+1}]} \quad (j \geqslant 1),$$

where $(a)_j = a(a+1)\cdots(a+j-1)$ denotes Pochhammer's symbol (see [10, 19]). A delicate analysis shows that $p_j \leqslant \frac{1}{2}$ ($j \in \mathbb{N}$) whenever $\alpha > 0$. By Theorem 3.1, the associated ultraspherical polynomials $P_j^{(\alpha)}(x,v)$ attain their absolute maxima (in the interval $[-1,1]$) at the points $+1$ and -1, provided that $\alpha > 0$.

Example 3.4 (*Sieved orthogonal polynomials on several intervals*). In [7], a system of sieved orthogonal polynomials on several intervals is considered. These polynomials are defined recursively by

$$\begin{aligned} &I_0(x,k) = 1, \qquad I_1(x,k) = x, \\ &(1+c)xI_m(x,k) = cI_{m+1} + I_{m-1}(x,k) \quad \text{if } m = nk \text{ for some } n \in \mathbb{N}, \qquad (9) \\ &2xI_m(x,k) = I_{m+1}(x,k) + I_{m-1}(x,k) \quad \text{else} \end{aligned}$$

($k \geqslant 2$ and $c > 0$) and are called sieved Chebyshev polynomials of the first kind. Let T_l and U_l denote the lth Chebyshev polynomial of the first and second kind, respectively. It is shown in [7] that for $c \geqslant 1$, the corresponding measure of orthogonality ξ_c is absolutely continuous with density

$$\frac{\mathrm{d}\xi_c}{\mathrm{d}x}(x) = \frac{\sqrt{4c - (1+c)^2 T_k^2(x)}}{2\pi(1-x^2)|U_{k-1}(x)|} I_{B_{k,c}}(x),$$

where

$$B_{k,c} = \left\{ y \in [-1,1] \,\Big|\, |T_k(y)| \leqslant \frac{2\sqrt{c}}{1+c} \right\}$$

and $I_{B_{k,c}}$ denotes the indicator function of the set $B_{k,c}$. If $c < 1$, ξ_c has two additional jumps of magnitude $(1-c)/2k$ at the points -1 and 1, and jumps of magnitude $(1-c)/k$ at the zeros of U_{k-1}. Calculating the monic orthogonal polynomials from (9) we obtain for the quantities p_j in (3) ($[a,b] = [-1,1]$)

$$p_j = \begin{cases} \dfrac{1}{c+1} & \text{if } j = 2kn \text{ for some } n \in \mathbb{N}, \\ \dfrac{1}{2} & \text{else}, \end{cases} \tag{10}$$

and Theorem 3.1 shows that for $c \geqslant 1$, the polynomials $I_j(x,k)$ in (9) attain their absolute maxima (in the interval $[-1,1]$) at the points -1 and 1. On the other hand, we have from [7, p. 99] that

$$I_{2k}(x,k) = \frac{1}{c}\left[(c+1)T_k^2(x) - 1\right].$$

Whenever $c < 1$, this polynomial attains its absolute maximum in the interval $[-1,1]$ at the zeros of kth Chebyshev polynomial of the first kind $T_k(x)$.

Corollary 3.5. *Let $\{\Phi_j\}_{j=0}^{\infty}$ denote a sequence of symmetric monic orthogonal polynomials defined by the recurrence relation ($\Phi_{-1} \equiv 0, \Phi_0 \equiv 1$)*

$$\Phi_{j+1}(x) = x\Phi_j(x) - \gamma_j \Phi_{j-1}(x) \quad (\gamma_j > 0, j \geqslant 0). \tag{11}$$

If for some $b > 0$,

$$\gamma_1 \leqslant \tfrac{1}{2}b^2, \qquad \gamma_j \leqslant \tfrac{1}{4}b^2 \quad (j \geqslant 2), \tag{12}$$

then the polynomials $\Phi_1, \Phi_2, \ldots$ attain their absolute maxima over the interval $[-b,b]$ at the points $-b$ and b.

Proof. The sequence $\{\gamma_j/b^2\}_{j=1}^{\infty}$ is a chain sequence (see [2, p. 97]) and by Theorem 2.1 in [2, p. 108], it follows that there exists a (symmetric) measure α on the interval $[-b,b]$ such that the polynomials $\{\Phi_j\}_{j=0}^{\infty}$ are orthogonal with respect to the measure $\mathrm{d}\alpha$. The quantities p_j in the continued fraction expansion (2) and recurrence relation (3) satisfy $p_{2j-1} = \frac{1}{2}$ (because of symmetry) and $q_{2j-2}p_{2j} = \gamma_j/b^2$ ($j \geqslant 1, q_0 = 1$) which means that $\{p_{2j}\}_{j=1}^{\infty}$ is the minimal parameter sequence of $\{\gamma_j/b^2\}_{j=1}^{\infty}$. From the assumption (12) we have $\gamma_j/b^2 \leqslant a_j$ ($j \in \mathbb{N}$) where

$$\{a_j\}_{j=1}^{\infty} = (\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \ldots)$$

is a chain sequence with minimal parameter sequence

$$\{m_j\}_{j=1}^{\infty} = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots)\,.$$

It is shown in the proof of Theorem 5.7 in [2, p. 97] that

$$p_{2j} \leqslant m_j = \tfrac{1}{2} \quad \forall j \in \mathbb{N}$$

and the assertion follows from Theorem 3.1 ($b = \tilde{b}$, $a = \tilde{a} = -b$). □

Remark 3.6. It can easily be seen that Corollary 3.5 can be generalized to the case where the $\{\gamma_j\}$ in the recurrence relation (11) satisfy inequality (12) only for the indices $j = 1, \ldots, n$. In this case, define

$$\begin{aligned} \tilde{\gamma}_j &= \gamma_j, \quad j = 1, \ldots, n\,, \\ \tilde{\gamma}_j &= \frac{b^2}{4}, \quad j \geqslant n+1\,, \end{aligned} \tag{13}$$

and monic orthogonal polynomials $\tilde{\Phi}_j(x)$ by the recurrence relation (11) with γ_j replaced by $\tilde{\gamma}_j$ ($j \geqslant 0$). By Corollary 3.5, it now follows that the polynomials $\{\tilde{\Phi}_j(x)\}$ attain their absolute maxima over the interval $[-b, b]$ at the points $-b$ and b. Observing (13), we thus obtain for the given monic orthogonal polynomials

$$\max_{x \in [-b,b]} |\Phi_j(x)| = \max_{x \in [-b,b]} |\tilde{\Phi}_j(x)| = |\tilde{\Phi}_j(\mp b)| = |\Phi_j(\mp b)| \tag{14}$$

whenever $j = 1, \ldots, n+1$. The same argument applies to polynomials orthogonal with respect to a measure with finite support (see Example 3.8).

Example 3.7 (*Associated Pollaczek and ultraspherical polynomials*). The Pollaczek polynomials were first studied in [15, 18]). Following [10], we define the (monic) associated Pollaczek polynomials by the recurrence relation

$$\Phi_{n+1}^{(v)}(x, \alpha, \mu) = x\Phi_n^{(v)}(x, \alpha, \mu) - \gamma_n \Phi_{n-1}^{(v)}(x, \alpha, \mu) \quad (n \geqslant 1)\,,$$

where $\alpha > 0$, $\mu \geqslant 0$, $v \geqslant 0$, $\Phi_0^{(v)}(x, \alpha, \mu) = 1$, $\Phi_1^{(v)}(x, \alpha, \mu) = x$ and

$$\gamma_n = \frac{(n+v)(n+v+2\alpha-1)}{4(n+v+\mu+\alpha)(n+v+\mu+\alpha-1)}\,. \tag{15}$$

It is now straightforward to show that $\gamma_1 \leqslant \frac{1}{2}$, $\gamma_j \leqslant \frac{1}{4}$ ($j \geqslant 2$) and by Corollary 3.5, the associated Pollaczek polynomial attains its absolute maximum in the interval $[-1, 1]$ at the points -1 and 1.

The choice $\mu = 0$ in (15) gives the monic associated ultraspherical polynomials while $\mu = 0$, $v = 0$ yields the common (monic) ultraspherical polynomials

$$\tilde{C}_{n+1}^{(\alpha)}(x) = x\tilde{C}_n^{(\alpha)}(x) - \frac{n(n+2\alpha-1)}{4(n+\alpha)(n+\alpha-1)}\,\tilde{C}_{n-1}^{(\alpha)}(x)$$

($\tilde{C}_0^{(\alpha)}(x) = 1$, $\tilde{C}_1^{(\alpha)}(x) = x$) which can also be defined for $-\frac{1}{2} < \alpha \leqslant 0$. From the previous discussion, we obtain the well-known result that $\tilde{C}_n^{(\alpha)}$ attains its absolute maximum in the interval $[-1, 1]$ at the points -1 and 1 whenever $\alpha \geqslant 0$. It is also known that this property does not hold any longer if $\alpha \in (-\frac{1}{2}, 0)$. In the sequel, we will apply Corollary 3.5 in order to determine an interval $[-b, b]$ such that $\tilde{C}_n^{(\alpha)}(x)$ is bounded on the interval $[-b, b]$ by its values at $-b$ and b. To this end let $\alpha \in (-\frac{1}{2}, 0)$ and

$$\gamma_n = \frac{n(n + 2\alpha - 1)}{4(n + \alpha)(n + \alpha - 1)} \quad (n \geqslant 1),$$

then $\gamma_1 \leqslant \frac{1}{2}b^2$ holds if $b \geqslant b_0 = (1 + \alpha)^{-1/2}$. On the other hand, if $n \geqslant 2$, then $\gamma_n \leqslant \frac{1}{4}b_0^2$ is equivalent to the inequality

$$n^2 - n(1 - 2\alpha) + 1 - \alpha \geqslant 0$$

(note that $\alpha < 0$) which is obviously satisfied for $n \geqslant 2$ and $-\frac{1}{2} < \alpha < 0$. Therefore, Corollary 3.5 shows that for $-\frac{1}{2} < \alpha < 0$, the ultraspherical polynomials $C_n^{(\alpha)}(x)$ attain their absolute maxima over the interval $[-(1 + \alpha)^{-1/2}, (1 + \alpha)^{-1/2}]$ at the points $\pm(1 + \alpha)^{-1/2}$.

Example 3.8 (*Symmetric Krawtchouk polynomials*). In a recent paper [4], it is shown that the (symmetric) Krawtchouk polynomials

$$k_n(x, N) = k_n\left(x, \frac{1}{2}, N\right) = \sum_{j=0}^{n} \frac{(-n)_j(-x)_j}{j!(-N)_j} 2^j$$

attain their absolute maxima over the interval $[0, N]$ at the points 0 and N whenever the degree n of $k_n(x, N)$ satisfies $n \leqslant \frac{1}{2}N + 1$. The corresponding monic polynomials orthogonal on the interval $[-\frac{1}{2}N, \frac{1}{2}N]$ with respect to the measure with masses $\binom{n}{k}(\frac{1}{2})^n$ at the points $\{-\frac{1}{2}N + k\}_{k=0}^{N}$ satisfy

$$\tilde{k}_{n+1}(x) = x\tilde{k}_n(x) - \left(\frac{N}{2}\right)^2 \frac{n}{N} \frac{(N - n + 1)}{N} \tilde{k}_{n-1}(x) \quad (n = 1, 2, \ldots, N - 1),$$

where $\tilde{k}_0(x) = 1$, $\tilde{k}_1(x) = x$. It is easy to see that the assumptions of Corollary 3.5 are satisfied with $b = \frac{1}{2}(N + 1)$ (note also Remark 3.6). Thus, a linear transformation yields that the symmetric Krawtchouk polynomials $k_1(x, N), \ldots, k_N(x, N)$ attain their absolute maxima over the interval $[-\frac{1}{2}, N + \frac{1}{2}]$ at the points $-\frac{1}{2}$ and $N + \frac{1}{2}$. Based on numerical results we conjecture that this property also holds on the "true" interval of orthogonality $[0, N]$.

Example 3.9 (*Tricomi–Carlitz polynomials*). The monic Tricomi–Carlitz polynomials are defined recursively by

$$F_{n+1}^{(\alpha)}(x) = xF_n^{(\alpha)}(x) - \frac{n}{(n + \alpha)(n + \alpha - 1)} F_{n-1}^{(\alpha)}(x) \quad (n \geqslant 1),$$

$F_0^{(\alpha)}(x) = 1$, $F_1^{(\alpha)}(x) = x$, $\alpha > 0$ (see [1], [2, p. 191]) and are orthogonal with respect to a step function whose jumps are

$$\frac{(k + \alpha)^{k-1}\mathrm{e}^{-k}}{k!} \quad \text{at} \quad x_k = \mp (k + \alpha)^{-1/2} \quad k = 0, 1, 2, \ldots .$$

It is easy to see that $F_n^{(\alpha)}$ does not attain its absolute maximum at the boundary of the true interval of orthogonality (e.g., $n = 3$, $\alpha = 1$, $F_3^{(1)}(x) = x^3 - \frac{5}{6}x$). However, if we investigate this property for the interval $[-1, 1]$, it follows from Corollary 3.5 that the Tricomi–Carlitz polynomials attain their absolute maxima at the points ∓ 1 whenever $\alpha \geqslant \frac{1}{2}(-3 + \sqrt{33}) \approx 1.3723$.

4. Christoffel functions

In this section, we investigate a different sufficient condition which is not directly related to the absolute maximum of the orthonormal polynomials $\{P_j(\mathrm{d}\alpha)\}_{j=0}^{n-1}$ in (4) and is used in the special case $[a,b] = [\tilde{a},\tilde{b}] = [-1,1]$ in [5] in order to study the G-efficiency of D-optimal designs in polynomial regression. The proof of the following result can be obtained by a similar reasoning as given in [5] using the identity (a) in Theorem 4.1 of [3], and is omitted for the sake of brevity.

Theorem 4.1. *Let α denote a probability measure on the interval $[a,b] \subseteq [\tilde{a},\tilde{b}]$, $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \ldots$ denote the quantities in the recurrence relation (5) for the monic orthogonal polynomials with respect to the measure* $\mathrm{d}\alpha$. *If*

$$\tilde{p}_{2j-1} \leqslant \tfrac{1}{2}, \quad j = 1, \ldots, n-1\,, \tag{16}$$

$$\frac{2\tilde{p}_{2j}-1}{1-\tilde{p}_{2j}}\,\frac{1-\tilde{p}_{2j+1}}{\tilde{p}_{2j+1}} \leqslant \frac{2\tilde{p}_{2j+2}-1}{\tilde{p}_{2j+2}}, \quad j = 1, \ldots, n-2\,, \tag{17}$$

then the Christoffel functions $\lambda_1(\mathrm{d}\alpha), \ldots, \lambda_n(\mathrm{d}\alpha)$ attain their minimum value in the interval $[\tilde{a},\tilde{b}]$ at the point $\tilde{b}$.

Remark 4.2. Replacing $\tilde{p}_{2j-1}$ by $1 - \tilde{p}_{2j-1}$ in inequalities (16) and (17) yield sufficient conditions such that the minimum of the Christoffel functions is attained at the point $\tilde{a}$ (see Remark 3.2).

Remark 4.3. If the sufficient conditions of Theorem 3.1 in Section 3 hold, then we have for the Christoffel functions defined in (4),

$$\lambda_n(\mathrm{d}\alpha, x) = \left[\sum_{j=0}^{n-1} P_j^2(\mathrm{d}\alpha, x)\right]^{-1} \geqslant \left[\sum_{j=0}^{n-1} P_j^2(\mathrm{d}\alpha, \tilde{b})\right]^{-1} = \lambda_n(\mathrm{d}\alpha, \tilde{b}) \tag{18}$$

whenever $x \in [\tilde{a},\tilde{b}]$. As an example, consider the measure corresponding to the ultraspherical polynomials $\mathrm{d}\mu(x) = (1-x^2)^{\alpha-1/2}\,\mathrm{d}x$. If $\alpha \geqslant 0$, we obtain from Example 3.7 and tedious algebra that

$$\lambda_n(\mathrm{d}\mu, x) \geqslant \frac{2^{2\alpha}\Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+\frac{3}{2})\Gamma(n)}{\Gamma(n+2\alpha)(n-\frac{1}{2}+\alpha)}$$

for all $x \in [-1, 1]$ with equality at the points -1 and 1. In the case of the Legendre weight ($\alpha = \frac{1}{2}$), we calculate the lower bound $2/n^2$ which improves the bound $1/8n^2$ usually found in the literature (see e.g. [13]).

Example 4.4 (*Discrete Chebyshev polynomials*). The discrete Chebyshev polynomials can be obtained from the well-known Hahn polynomials

$$Q_n(x,\alpha,\beta,N) = \sum_{k=0}^{n} \frac{(-n)_k(n+\alpha+\beta+1)_k(-x)_k}{k!(\alpha+1)_k(-N)_k}$$

(see [8]) by putting $\alpha = \beta = -\frac{1}{2}$, i.e.,

$$T_n^*(x) = Q_n(x, -\tfrac{1}{2}, -\tfrac{1}{2}, N) .$$

The polynomials $T_n^*(x)$ are orthogonal with respect to the jump function

$$\varphi_N^*(x) = \binom{x-\frac{1}{2}}{x}\binom{N-x-\frac{1}{2}}{N-x}, \quad x = 0,\dots,N \tag{19}$$

[here $\binom{a}{k} = a(a-1)\cdots(a-k+1)/k!$ denotes the binomial coefficient] and satisfy the recurrence relation

$$T_0^*(x) = 1, \qquad T_1^*(x) = 1 - \frac{2}{N}x ,$$

$$\left(\frac{N}{2} - x\right) T_n^*(x) = \frac{n+N}{4} T_{n-1}^*(x) + \frac{N-n}{4} T_{n+1}^*(x) ,$$

$n = 0,\dots,N-1$ (see [8]). Note that in general $T_n^*(x)$ does not attain its absolute maximum at the point 0 or N (see [20, p. 344]). Considering the monic form of these polynomials, we obtain for the quantities p_j in (3) ($[a,b] = [0,N]$),

$$p_{2j} = \frac{N+j}{2N}, \quad p_{2j-1} = \frac{1}{2}, \qquad j = 1,\dots,N. \tag{20}$$

In Theorem 4.1, we put $a = \tilde{a} = 0$, $b = \tilde{b} = N$ and $\tilde{p}_j = p_j$ ($j \in \mathbb{N}$) (note (7)). Inequality (16) is obviously satisfied and (17) yields

$$\frac{2j}{N-j} \leqslant \frac{2(j+1)}{N+j+1},$$

which holds for all $j \leqslant \frac{1}{2}(-1+\sqrt{2N+1})$. By Theorem 4.1, the Christoffel functions $\lambda_1(d\varphi_N^*),\dots,\lambda_n(d\varphi_N^*)$ attain their minima (over the interval $[0,N]$) at the points 0 and N whenever $n \leqslant n_0 = \frac{1}{2}(3+\sqrt{2N+1})$. As an example that this bound cannot be improved, consider the case $N = 2$ which yields $n_0 = \frac{1}{2}(3+\sqrt{5})$. The third Christoffel function is given by

$$\lambda_3(d\varphi_2^*, x) = [1 + \tfrac{4}{3}(x-1)^2 + \tfrac{16}{3}(x^2 - 2x + \tfrac{1}{4})^2]^{-1}$$

and its minimum over the interval $[0,2]$ is attained at the point 1, i.e., $\lambda_3(d\varphi_2^*, 1) = \frac{1}{4} < \frac{3}{8} = \lambda_3(d\varphi_2^*, 0) = \lambda_3(d\varphi_2^*, 2)$.

Example 4.5 (*Equality in Theorem* 4.1). Consider the uniform distribution ξ^* on the set

$$\{x \mid (1-x^2)P_n'(x) = 0\} , \tag{21}$$

where P_n' denotes the derivative of the nth Legendre polynomial. Let $x_0^*, \dots, x_n^*$ denote the zeros of the polynomial $(1-x^2)P_n'(x)$. Then it is well known that the tuple $(x_0^*, \dots, x_n^*)$ is the unique solution of the following extremal problem:

Let $\mathcal{M}_{n+1}$ denote the set of all $(n+1)$-tuples $(x_0, \dots, x_n)$ with $-1 \leqslant x_0 \leqslant \cdots \leqslant x_n \leqslant 1$ and let $L_i(x, x_0, \dots, x_n)$ denote the ith Lagrange interpolatory polynomial with nodes $x_0, \dots, x_n$ $(i = 0, \dots, n)$. Find the tuple $(x_0^*, \dots, x_n^*)$ which minimizes

$$\sup_{x \in [-1,1]} \sum_{j=0}^{n} L_j^2(x, x_0, \dots, x_n) \tag{22}$$

over all tuples $(x_0, \dots, x_n) \in \mathcal{M}_{n+1}$.

The minimal value M of (22) is attained for the tuple $(x_0^*, \dots, x_n^*)$ and given by $M = 1$ (see [6, 9] or [12]). If ξ is a uniform distribution with $n+1$ support points $x_0, \dots, x_n$, then the $(n+1)$st $\lambda_{n+1}(\mathrm{d}\xi)$ Christoffel function can be represented as

$$\lambda_{n+1}(\mathrm{d}\xi, x) = \left[(n+1) \sum_{j=0}^{n} L_j^2(x, x_0, \dots, x_n) \right]^{-1}.$$

Consequently, the Christoffel function corresponding to the uniform distribution ξ^* on the set (21) satisfies

$$\min_{x \in [-1,1]} \lambda_{n+1}(\mathrm{d}\xi^*, x) = \left[(n+1) \max_{x \in [-1,1]} \sum_{j=0}^{n} L_j^2(x, x_0^*, \dots, x_n^*) \right]^{-1} = \frac{1}{n+1}$$

with equality for $x = x_j^*$ $(j = 0, \dots, n)$. Thus, the measure ξ^* gives an extreme case in the sense that its corresponding Christoffel function $\lambda_{n+1}(\mathrm{d}\xi^*)$ has $n+1$ global minima in the interval $[-1, 1]$ at the support points $x_0^*, \dots, x_n^*$ of ξ^* defined by (21). This extremal property is also reflected in the quantities $(p_j^*)_{j \in \mathbb{N}}$ in the continued fraction expansion (2) for the Stieltjes transform of ξ^*. In fact, it is shown in [11] that

$$p_{2j-1}^* = \frac{1}{2}, \quad p_{2j}^* = \frac{n-j+1}{2(n-j)+1}, \qquad j = 1, \dots, n \tag{23}$$

(note the minor mistake in formula (9) of this reference), and straightforward calculations show that the p_j^* defined by (23) satisfy the assumptions (16) and (17) of Theorem 4.1 with equality for all $j = 1, \dots, n-2$.

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References

[1] L. Carlitz, On some polynomials of Tricomi, *Boll. Un. Mat. Ital.* **13** (1958) 58–64.
[2] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).
[3] H. Dette, New identities for orthogonal polynomials on a compact interval, *J. Math. Anal. Appl.* **179** (1993) 547–573.
[4] H. Dette, New bounds for Hahn and Krawtchouk polynomials, *SIAM J. Math. Anal.*, to appear.
[5] H. Dette and W.K. Wong, On G-efficiency calculation for polynomial models, *Ann. Statist.*, to appear.
[6] L. Fejér, Bestimmung derjenigen Abzissen eines Intervalls, für welche die Quadratsumme der Grundfunktionen der Lagrangeschen Interpolation im Intervall ein möglichst kleines Maximum besitzt, *Ann. Scn. Norm. Sup. (Pisa)* **1** (1932) 3–16.
[7] M.E.H. Ismail, On sieved orthogonal polynomials III: Orthogonality on several intervals, *Trans. Amer. Math. Soc.* **294** (1986) 89–111.
[8] S. Karlin and J. McGregor, The Hahn polynomials, formulas and an application, *Scripta Math.* **26** (1961) 33–46.
[9] S. Karlin and W.J. Studden, *Tchebycheff Systems: With Applications in Analysis and Statistics* (Interscience, New York, 1966).
[10] R. Lasser, Orthogonal polynomials on hypergroups II — the symmetric case, *Trans. Amer. Math. Soc.* **341** (1994) 749–771.
[11] T.S. Lau and W.J. Studden, Optimal designs for trigonometric and polynomial regression using canonical moments, *Ann. Statist.* **13** (1985) 383–394.
[12] T.S. Lau and W.J. Studden, On an extremal problem of Fejér, *J. Approx. Theory* **53** (1988) 184–194.
[13] A.L. Levin and D.S. Lubinsky, Christoffel functions, orthogonal polynomials and Nevai's conjecture, *Constr. Approx.* **8** (1992) 463–535.
[14] P. Nevai, Géza Freud, orthogonal polynomials and Christoffel functions. A case study, *J. Approx. Theory* **48** (1986) 3–167.
[15] F. Pollaczek, Sur une famille de polynomes orthogonoux à quatre paramatres, *C.R. Acad. Sci. Paris* **230** (1950) 2254–2256.
[16] M. Skibinsky, Some striking properties of binomial and beta moments, *Ann. Math. Statist.* **40** (1969) 1753–1764.
[17] M. Skibinsky, Principal representations and canonical moment sequences for distributions on an interval, *J. Math. Anal. Appl.* **120** (1986) 95–120.
[18] G. Szegő, On certain special sets of orthogonal polynomials, *Proc. Amer. Math. Soc.* **1** (1950) 731–737.
[19] J. Wimp, Explicit formulas for the associated Jacobi polynomials, *Can. J. Math.* **39** (1987) 983–1000.
[20] S.K. Zaremba, Some properties of polynomials orthogonal over the set $\langle 1, 2, \ldots, N \rangle$, *Ann. Math. Pura. Appl.* **105** (1975) 333–345.